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Creation of Scalar and Dirac Particles in Asymptotically Flat Robertson-Walker Spacetimes

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Abstract In the present article we obtain the exact solutions of the Klein-Gordon and Dirac equations for two models of Robertson-Walker spaces with asymptotically Minkowskian regions. Using the obtained exact solutions we calculate the density of scalar and Dirac particles created through Bogolubov transformations technique. For Dirac field it is shown that the creation rate of particles and anti particles are equal.

Keywords Particle creation · Bogolubov transformation

1 Introduction

Particle creation as a consequence of expansion of the universe, which was first discussed by Parker [1–4], is the most interesting result of quantum field theory in curved spacetime. The calculation of particle production rate in strong gravitational fields has the considerable results in cosmology and astrophysics [1–7]. It should be noted that at the present time, because of the very small rate of expansion, the particle production is almost zero but at the early stage of universe was much larger than at the present. One of the interesting results of particle production in an expanding universe is that the particles are created in pairs with equal amount of matter and antimatter, which is a favorable result in cosmology. Such a semiclassical treatment for gravity could be useful for understanding a theory which unifies gravity and quantum mechanics. Almost forty years ago Parker discovered that the expansion of the universe can create pairs of particles. He showed that fields obeying conformally invariant wave equations will not be produced, because the FRW models are conformally flat. When the conformal symmetry of the FRW models broken by anisotropy, conformally invariant particles will be produced.

In Minkowskian spacetime plane waves are eigenfunctions of the timelike killing vector. The vacuum defined by these natural basis is invariant under the Poincarè group. In a curved spacetime that does not possess the timelike killing vector to define the positive frequency

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modes, there is not a preferred vacuum state. In some spacetimes with asymptotically static regions there may exist natural basis analogues Minkowskian space associated with the timelike killing vector to define the vacuum state. Because of the time dependence of the metric in the intermediate region, positive frequency solution in the *in* region (remote past) will not correspond to a purely positive frequency solution in the *out* region (far future), then the vacuum in the remote past is different from the one in the remote future. This difference causes the frequency mixing and therefore the particle creation. In other words, the expansion of the universe will result in spontaneous particle creation. The rate of particle creation can be computed through Bogolubov transformations, which requires first to identify the *in* and *out* vacuum states. For a general class of spacetimes, to find the solution of equations and then the Bogolubov coefficients are extremely difficult.

The paper is organized as follows. In Sect. 2 we present a brief discussion on Bogolubov transformations for scalar and Dirac fields. In Sect. 3 we solve the Klein-Gordon and Dirac equations for two types of Robertson-Walker spacetimes. One model is asymptotically Minkowskian in the *in* and *out* regions. In this case we obtain the analytic solutions of massive conformally coupled scalar field and particle creation rate. In another model which is Minkowskian in the *in* region we solve massive conformally and minimally coupled scalar field and analyze the particle creation rate. Also we obtain the exact solutions of Dirac equation and creation rate of particles and antiparticles. We conclude with a discussion in Sect. 4.

2 Bogolubov Transformations

2.1 Scalar Field

Assume that φ^{in} and φ^{out} are positive frequency solutions of Klein-Gordon equation in the remote past and far future. Each set is orthonormal so that

$$\begin{aligned} (\varphi_k^{in}, \varphi_{k'}^{in}) = (\varphi_k^{out}, \varphi_{k'}^{out}) = \delta_{kk'}, \\ (\varphi_k^{in}, \varphi_{k'}^{*in}) = (\varphi_k^{out}, \varphi_{k'}^{*out}) = 0. \end{aligned} \tag{1}$$

Although these functions are defined by their asymptotic properties in different regions they are solutions of the wave equation every where in spacetime. As both sets are complete, we can expand *in*-modes in terms of *out*-modes

$$\varphi_j^{in} = \sum_k \alpha_{jk}\varphi_k^{out} + \beta_{jk}\varphi_k^{*out}, \tag{2}$$

where the Bogolubov coefficients α_{jk}, β_{jk} possess the following properties

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij}, \tag{3}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0. \tag{4}$$

Bogolubov coefficients relate the *in* and *out* operators and give directly the number of particles created, i.e., the number of *out* particles contained in the *in* vacua. Two sets of creation and annihilation operators can be expanded in terms of one another by Bogolubov transformations

$$a_k^{out} = \sum_j (\alpha_{jk} a_j^{in} + \beta_{jk}^* a_j^{in\dagger}), \tag{5}$$

$$a_j^{in} = \sum_k (\alpha_{jk}^* a_k^{out} - \beta_{jk}^* a_k^{out\dagger}). \tag{6}$$

There are two different vacuum $|0_{in}\rangle$ and $|0_{out}\rangle$ associated with two Fock spaces $\mathcal{F}^{in}$ and $\mathcal{F}^{out}$

$$\begin{aligned} a_k^{in}|0_{in}\rangle &= 0, \quad \forall k, \\ a_k^{out}|0_{out}\rangle &= 0, \quad \forall k. \end{aligned} \tag{7}$$

The expectation value of number operator for the *out*-modes in the state $|0_{in}\rangle$ is

$$\langle N_k \rangle = \langle 0_{in}| a_k^{out\dagger} a_k^{out} |0_{in}\rangle = \sum_j |\beta_{jk}|^2. \tag{8}$$

Therefore an initially vacuum state in far past acquires at remote future a background of massive scalar particles.

In a spatially flat spacetime because of spatial translation symmetry, the spatial modes are orthogonal, so the Bogolubov coefficients will be diagonal in k

$$\alpha_{\mathbf{k}\mathbf{k}'} = \alpha_k \delta_{\mathbf{k}\mathbf{k}'}, \qquad \beta_{\mathbf{k}\mathbf{k}'} = \beta_k \delta_{-\mathbf{k}\mathbf{k}'}, \tag{9}$$

i.e. Bogolubov coefficients mix only modes of wave vectors $\mathbf{k}$ and $-\mathbf{k}$, and they depend only the magnitude of the $\mathbf{k}$. Now the Bogolubov transformation (2) reduces to

$$\varphi_{\mathbf{k}}^{in} = \alpha_k \varphi_{\mathbf{k}}^{out} + \beta_k \varphi_{-\mathbf{k}}^{*out}. \tag{10}$$

Taking into account normalization condition for scalar field $|\alpha_k|^2 - |\beta_k|^2 = 1$, the average number of particles created will be

$$\langle 0_{in}| N_{\mathbf{k}}^{out} |0_{in}\rangle = n_k = \frac{\gamma_k}{1-\gamma_k}, \tag{11}$$

where $\gamma_k = |\beta_k|^2/|\alpha_k|^2$.

2.2 Dirac Field

The Dirac field operator can be written as

$$\psi = \sum_i a_i u_i(x) + b_i^\dagger v_i(x), \tag{12}$$

where sum indicates an integral over momenta and sum over polarizations. The a_i and b_i are respectively annihilation of particles and antiparticles. Let assume that $\{u_i^{in}(x), v_i^{in}(x)\}$ and $\{u_i^{out}(x), v_i^{out}(x)\}$ are two complete set of mode solutions of Dirac equation which define particles and antiparticles in asymptotic regions. Since each set is complete we can write down one set in terms of another

$$u_i^{in} = \sum_j \alpha_{ij} u_j^{out} + \beta_{ij} v_j^{out}, \tag{13}$$

$$v_i^{in} = \sum_j \varrho_{ij} u_j^{out} + \sigma_{ij} v_j^{out}. \tag{14}$$

Then the *in* and *out* creation operator of particles and antiparticles are related by

$$a_i^{out} = \sum_j \alpha_{ij} a_j^{in} + \varrho_{ij} b_j^{in\dagger}, \tag{15}$$

$$b_i^{out} = \sum_j \beta_{ij}^* a_j^{in\dagger} + \sigma_{ij} b_j^{in}. \tag{16}$$

The expectation value of the *out* particles in the *in* vacuum (i.e., particles created) is

$$\langle N_j^p \rangle = \langle 0_{in} | a_j^{out\dagger} a_j^{out} | 0_{in} \rangle = \sum_i |\varrho_{ij}|^2. \tag{17}$$

Similarly the expectation value of *out* antiparticles in the *in* vacuum (i.e., antiparticles created) is

$$\langle N_j^a \rangle = \langle 0_{in} | b_j^{out\dagger} b_j^{out} | 0_{in} \rangle = \sum_i |\beta_{ij}|^2. \tag{18}$$

Then total number of particles created is

$$N = \sum_i |\beta_{ij}|^2 + |\varrho_{ij}|^2. \tag{19}$$

3 Particle Creation

3.1 Scalar Particles

Klein-Gordon equation in curved spacetime is [5]

$$[\Box_x + m^2 + \xi R(x)]\varphi(x) = 0, \tag{20}$$

where ξ is a constant and R is the scalar curvature, $\xi R(x)$ is necessary to make the equation conformally invariant in the massless case. One must set ξ equal to $\frac{1}{6}$ in order to have that property, in this case we say there is conformal coupling to gravitational field. If $\xi = 0$ we say there is minimal coupling. Consider a spatially flat Robertson-Walker space with the metric

$$ds^2 = dt^2 - a^2(t) dx_i dx^i, \tag{21}$$

in terms of conformal time parameter η given by $\eta = \int dt/a(t)$, the line element (21) to be

$$ds^2 = C(\eta)(d\eta^2 - dx_i dx^i), \tag{22}$$

where $C(\eta) = a^2(\eta) = b^s(\eta)$. Since this form of the line element is conformal to Minkowskian space, one has a correspondence between fields in a Robertson-Walker spacetime and in flat spacetime with time dependent mass. Because of the spatial homogeneity, we can separate the mode solutions of the wave equation

$$\varphi(x) = (2\pi)^{-3/2}C^{-1/2}\varphi_{\mathbf{k}}(\eta, x) = (2\pi)^{-3/2}C^{-1/2}e^{i\mathbf{k}\cdot\mathbf{x}}\chi_k(\eta), \tag{23}$$

where $k = |\mathbf{k}|$. Then the equation satisfies by $\chi_k(\eta)$ is

$$\left[\frac{d^2}{d\eta^2} + k^2 + b^s(m^2 + (\xi - 1/6)R)\right]\chi_k(\eta) = 0 \tag{24}$$

where

$$R(\eta) = 3sb^{-s}\left[(s/2-1)\frac{\dot{b}^2}{b^2} + \frac{\ddot{b}}{b}\right]. \tag{25}$$

Equation (24) possesses the formal WKB-type solution [5]

$$\chi_k = (2W_k)^{-\frac{1}{2}}\exp\left[-i\int^{\eta} W_k(\eta')d\eta'\right], \tag{26}$$

where W_k satisfies the nonlinear equation

$$W_k^2(\eta) = \omega_k^2(\eta) - \frac{1}{2}\left(\frac{\ddot{W}_k}{W_k} - \frac{3}{2}\frac{\dot{W}_k^2}{W_k^2}\right), \tag{27}$$

with

$$\omega_k^2(\eta) = k^2 + b^s(m^2 + (\xi - 1/6)R).$$

If the spacetime is slowly varying, then the derivative terms in (27) will be small compared to $\omega_k^2(\eta)$, so a zeroth order approximation is to substitute

$$W_k^{(0)} = \omega_k(\eta) \tag{28}$$

into the integrand of (26). This solution obviously reduces to the standard Minkowskian spacetime modes as $C(\eta) \to$ constant. Suppose that instead of using the exact solution χ_k given by (26) one use the zeroth order adiabatic approximation obtained by replacing W_k by $W_k^{(0)}$. Here we consider to some exactly soluble models. One model is

$$C(\eta) = \left(\frac{1+\delta}{2} + \frac{1-\delta}{2}\tanh(\lambda\eta)\right)^s, \quad s, \delta, \lambda = \text{constants}, \tag{29}$$

which is Minkowskian in the far past and future, i.e., $C(\eta) \to \delta^s$ in the *in* region and $C(\eta) \to 1$ at the *out* region. Case $s = 1$ was first discussed by Bernard and Duncan [11]. Note that this spacetime has no big bang, instead the universe starts out as Minkowskian space, then expands smoothly and ends up as another Minkowskian space. It is interesting to analyze this universe in the limit $\delta \to -1$ and $0 < \eta \ll 1/\lambda$. For $s = 2$, $a(t) \propto t^{1/2}$, so universe behaves like radiation-dominated Friedman cosmology. In similar situation for $s = -2$, the universe approaches to de Sitter model with $a(t) \propto e^{\lambda t}$.

For massive conformally coupled case and scale factor (29) equation (24), reduces to

$$\left[\frac{d^2}{d\eta^2}+k^2+\left(\frac{1+\delta}{2}+\frac{1-\delta}{2}\tanh(\lambda\eta)\right)^s m^2\right]\chi_k(\eta)=0. \tag{30}$$

This equation can be solved in terms of hypergeometric functions for $s=\pm1,\pm2$. We use the method of Bogolubov transformations to compute the density of particles created. To apply this method it is necessary to specify positive modes in the remote past and future. After some algebra, the solutions of (30) behaving as positive frequency modes as $\eta\to-\infty$ $(t\to-\infty)$, are found to be

$$\chi_k^{in}(\eta) = \begin{cases} (2\Omega_{in})^{-1/2}\exp[-i\Omega_+\eta-\frac{i\Omega_-}{\lambda}\ln[2\cosh\lambda\eta]] \\ \quad\times F(\frac{1}{2}-\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},\frac{1}{2}+\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},1-i\frac{\Omega_{in}}{\lambda},\frac{1+\tanh(\lambda\eta)}{2}), & s=1,2, \\ (2\Omega_{in})^{-1/2}\exp[-i\Omega_+\eta-\frac{i\Omega_-}{\lambda}\ln(2b(\eta)\delta^{-1}\cosh\lambda\eta)] \\ \quad\times F(\frac{1}{2}-\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},\frac{1}{2}+\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},1-i\frac{\Omega_{in}}{\lambda},\frac{\delta(1+\tanh(\lambda\eta))}{1+\delta+(1-\delta)\tanh(\lambda\eta)}), & s=-1,-2, \end{cases} \tag{31}$$

where

$$\Omega_{in}=[k^2+m^2\delta^s]^{\frac{1}{2}},\qquad \Omega_{out}=[k^2+m^2]^{\frac{1}{2}}, \tag{32}$$

$$\Omega_{in}=\Omega_+-\Omega_-,\qquad \Omega_{out}=\Omega_++\Omega_-,\qquad \Omega_\pm=\frac{1}{2}(\Omega_{out}\pm\Omega_{in}), \tag{33}$$

$$\bar\Omega_{\pm1}=i\lambda,\qquad \bar\Omega_{\pm2}=[m^2(1-\delta^{\pm1})^2-\lambda^2]^{1/2}. \tag{34}$$

In the *in* region when $\eta\to-\infty$, these modes reduce to Minkowskian exponential modes

$$\chi_k^{in}(\eta)\to(2\Omega_{in})^{-1/2}e^{-i\Omega_{in}\eta}. \tag{35}$$

However in the *out* region, where $\eta\to+\infty$, these modes are not simple exponentials, but are more complicated functions of time. Similarly, one may find a complete set of modes of the field that behaving as positive frequency modes as $\eta\to+\infty$ $(t\to+\infty)$

$$\chi_k^{out}(\eta) = \begin{cases} (2\Omega_{out})^{-1/2}\exp[-i\Omega_+\eta-\frac{i\Omega_-}{\lambda}\ln[2\cosh\lambda\eta]] \\ \quad\times F(\frac{1}{2}-\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},\frac{1}{2}+\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},1+i\frac{\Omega_{out}}{\lambda},\frac{1-\tanh(\lambda\eta)}{2}), & s=1,2, \\ (2\Omega_{out})^{-1/2}\exp[-i\Omega_+\eta-\frac{i\Omega_-}{\lambda}\ln(2b(\eta)\cosh\lambda\eta)] \\ \quad\times F(\frac{1}{2}-\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},\frac{1}{2}+\frac{i\bar\Omega_s}{2\lambda}+\frac{i\Omega_-}{\lambda},1+i\frac{\Omega_{out}}{\lambda},\frac{1-\tanh(\lambda\eta)}{1+\delta+(1-\delta)\tanh(\lambda\eta)}), & s=-1,-2. \end{cases} \tag{36}$$

Now we are ready to calculate the particle creation rate by the expanding universe. Using the linear transformation properties of hypergeometric functions [12]

$$F(a,b,c,z)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}F(a,b,a+b-c+1,1-z)$$

$$+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \times F(c-a, c-b, c-a-b+1, 1-z), \quad (37)$$

and

$$F(a,b,c;z) = (1-z)^{c-a-b}F(c-a, c-b, c; z), \quad (38)$$

we can evaluate Bogolubov transformation between $\chi_k^{in}(\eta)$ and $\chi_k^{out}(\eta)$

$$\chi_k^{in}(\eta) = \alpha_k\chi_k^{out}(\eta) + \beta_k\chi_k^{out*}(\eta), \quad (39)$$

where the Bogolubov coefficients are given by

$$\alpha_k = \left(\frac{\Omega_{out}}{\Omega_{in}}\right)^{1/2} \frac{\Gamma(1-i\frac{\Omega_{in}}{\lambda})\Gamma(-i\frac{\Omega_{out}}{\lambda})}{\Gamma(\frac{1}{2}+\frac{i\bar{\Omega}_s}{2\lambda}-\frac{i\Omega_+}{\lambda})\Gamma(\frac{1}{2}-\frac{i\bar{\Omega}_s}{2\lambda}-\frac{i\Omega_+}{\lambda})}, \quad (40)$$

$$\beta_k = \left(\frac{\Omega_{out}}{\Omega_{in}}\right)^{1/2} \frac{\Gamma(1-\frac{i\Omega_in}{\lambda})\Gamma(i\frac{\Omega_{out}}{\lambda})}{\Gamma(\frac{1}{2}-\frac{i\bar{\Omega}_s}{2\lambda}+\frac{i\Omega_-}{\lambda})\Gamma(\frac{1}{2}+\frac{i\bar{\Omega}_s}{2\lambda}+\frac{i\Omega_-}{\lambda})}. \quad (41)$$

Using the following relations [12]

$$|\Gamma(iy)|^2 = \frac{\pi}{y\sinh\pi y}, \quad (42)$$

$$|\Gamma(1+iy)|^2 = \frac{\pi y}{\sinh\pi y}, \quad (43)$$

$$\left|\Gamma\left(\frac{1}{2}+iy\right)\right|^2 = \frac{\pi}{\cosh\pi y}, \quad (44)$$

the density of particles created reads

$$n_k = \frac{\cosh\pi(\frac{\bar{\Omega}_s}{\lambda}) + \cosh\pi(\frac{2\Omega_-}{\lambda})}{\cosh\pi(\frac{2\Omega_+}{\lambda}) - \cosh\pi(\frac{2\Omega_-}{\lambda})}. \quad (45)$$

In the massless case limit, $n_k \to 0$: no particle production occurs, i.e., particle creation occurs when the conformal symmetry is broken by mass, on the other hand particle production is as a consequence of coupling of the spacetime expansion to the quantum field via the mass. It is obvious from (45), the rate of particles creation depends on the expansion rate. In the limit weak expansion, i.e., $\lambda \to 0$, creation rate approaches to zero. For high mass particles the particle creation rate reduces to

$$n_k \approx e^{-\frac{2\pi m}{\lambda}\delta^{s/2}}, \quad 0 < \delta \le 1, \quad (46)$$

this is the Planck spectrum for radiation at temperature $T = \frac{\lambda}{2\pi\delta^{s/2}}$. The creation of high mass particles for $m \gg \lambda$, approaches zero sharply, which is reasonable, because production of high energy particles needs much more changing of gravitational field.

Another soluble model is given by the scale factor

$$C(\eta) = (1+e^{\lambda\eta})^s, \quad (47)$$

which has an asymptotically static *in* region

$$C(\eta) \to 1, \quad \eta \to -\infty. \tag{48}$$

There are analytic solutions for conformally coupled scalar field in cases $s = \pm 1, \pm 2$, but we consider only to the case $s = -2$. This case is interesting for us because we can obtain exact solutions of massive minimally and conformally coupled scalar field as well as Dirac field. Then we can analyze particle creation for scalar and Dirac fields in this case. In this spacetime we have well defined vacuum state in the *in* region because in this region spacetime is Minkowskian. But in the *out* region we have to use WKB approximation for obtaining positive and negative modes. In this model, cosmological time t related to conformal time η by

$$t = \eta + \frac{e^{\lambda\eta}}{\lambda}, \tag{49}$$

then asymptotic values $\eta \to \pm\infty$ correspond to $t \to \pm\infty$. The scalar curvature of this spacetime is

$$R = 6\lambda^2 e^{\lambda\eta}(e^{\lambda\eta} - 1). \tag{50}$$

One may write down a complete set of field modes that reduce to standard positive frequency modes in the *in* region when $\eta \to -\infty$ ($t \to -\infty$)

$$\begin{aligned}\chi_k^{in}(\eta) = (2\omega)^{-1/2} \exp\left[-i\omega\eta + \frac{1}{2}\left(1 - \frac{i\bar{\omega}}{\lambda}\right)\ln[1 + e^{\lambda\eta}]\right] \\ \times F\left(\frac{1}{2} - \frac{i}{\lambda}\left(\omega + \hat{\omega} + \frac{\bar{\omega}}{2}\right), \frac{1}{2} - \frac{i}{\lambda}\left(\omega - \hat{\omega} + \frac{\bar{\omega}}{2}\right), 1 - \frac{2i\omega}{\lambda}, -e^{\lambda\eta}\right),\end{aligned} \tag{51}$$

where

$$\omega = [k^2 + m^2]^{\frac{1}{2}}, \tag{52}$$

$$\bar{\omega} = (4m^2 + 48\lambda^2\xi - 9\lambda^2)^{\frac{1}{2}}, \tag{53}$$

$$\hat{\omega} = (k^2 + 6\lambda^2\xi - \lambda^2)^{\frac{1}{2}}, \tag{54}$$

in this limit the (51) reduces to

$$\chi_k(\eta) \sim (2\omega)^{-\frac{1}{2}} e^{-i\omega\eta}, \tag{55}$$

which is positive frequency Minkowskian mode. In the out region we use the WKB approximation to determine the positive frequency modes. The zeroth order adiabatic solution in the asymptotic regions we have

$$\chi_k^{(0)} \approx \begin{cases} (2\hat{\omega})^{-\frac{1}{2}} e^{-i\hat{\omega}\eta}, & \eta \to +\infty, \\ (2\omega)^{-\frac{1}{2}} e^{-i\omega\eta}, & \eta \to -\infty. \end{cases} \tag{56}$$

Now we are ready to calculate the Bogolubov coefficients. Using the linear transformation properties of hypergeometric functions [12]

$$F(a, b, c, z) = \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b} F\left(b, 1 - c + b, 1 - a + b; \frac{1}{z}\right)$$

$$+\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}F\left(a,1-c+a,1-b+a;\frac{1}{z}\right), \tag{57}$$

when $z\to+\infty$

$$F(a,b,c,z)\approx\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}+\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}, \tag{58}$$

the Bogolubov coefficients are given by

$$\alpha_k=\left(\frac{k}{\omega}\right)^{1/2}\frac{\Gamma(1-2i\frac{\omega}{\lambda})\Gamma(-2\frac{i\hat{\omega}}{\lambda})}{\Gamma(\frac{1}{2}-\frac{i}{\lambda}(\omega+\hat{\omega}+\frac{\bar{\omega}}{2}))\Gamma(\frac{1}{2}-\frac{i}{\lambda}(\omega+\hat{\omega}-\frac{\bar{\omega}}{2}))}, \tag{59}$$

$$\beta_k=\left(\frac{k}{\omega}\right)^{1/2}\frac{\Gamma(1-2i\frac{\omega}{\lambda})\Gamma(2\frac{i\hat{\omega}}{\lambda})}{\Gamma(\frac{1}{2}-\frac{i}{\lambda}(\omega-\hat{\omega}+\frac{\bar{\omega}}{2}))\Gamma(\frac{1}{2}-\frac{i}{\lambda}(\omega-\hat{\omega}-\frac{\bar{\omega}}{2}))}. \tag{60}$$

After some mathematical manipulation the density of particles created reads

$$n_k=\frac{\cosh(\frac{\pi\bar{\omega}}{\lambda})+\cosh\frac{2\pi}{\lambda}[\omega-\hat{\omega}]}{\cosh\frac{2\pi}{\lambda}(\omega+\hat{\omega})-\cosh\frac{2\pi}{\lambda}(\omega-\hat{\omega})}. \tag{61}$$

In the massless case we have

$$\omega=k,\qquad \bar{\omega}=(48\lambda^2\xi-9\lambda^2)^{\frac{1}{2}},\qquad \hat{\omega}=(k^2+6\lambda^2\xi-\lambda^2)^{\frac{1}{2}}. \tag{62}$$

It's obvious in conformally coupled case as $\xi=\frac{1}{6}$, particle creation rate is zero. But in minimally coupled case as $\xi=0$ we have

$$n_k=\frac{\cosh\frac{2\pi}{\lambda}(k-\hat{\omega})-1}{\cosh\frac{2\pi}{\lambda}(k+\hat{\omega})-\cosh\frac{2\pi}{\lambda}(k-\hat{\omega})}. \tag{63}$$

3.2 Dirac Particles

The behavior of relativistic particles obeying the Dirac equation in curved spaces, in particular in expanding universe is of considerable importance in astrophysics and cosmology [8–10]. One of the most important but difficult problems of the quantum field theory in curved spacetime is the problem of finding analytic solutions of Dirac equation on given backgrounds. Barut and Duru investigated the exact solutions of the Dirac equation for three typical models of expanding universe [8]. The Dirac equation in the line element (21) becomes

$$\left(i\gamma^\mu\partial_\mu+i\frac{3}{2}\frac{\dot{a}}{a}\gamma^0-ma\right)\Psi=0, \tag{64}$$

where dot refers to differentiation with respect to conformal time η and flat-Dirac matrices are defined as follows

$$\gamma^0=\begin{pmatrix}I&0\\0&-I\end{pmatrix},\qquad \gamma^i=\gamma^0\alpha^i,\quad \alpha^i=\begin{pmatrix}0&\sigma^i\\\sigma^i&0\end{pmatrix}. \tag{65}$$

With substituting

$$\Psi=(2\pi)^{-3/2}e^{i\mathbf{k}\cdot\mathbf{x}}a^{-3/2}\begin{pmatrix}\Phi_1\\\Phi_2\end{pmatrix}, \tag{66}$$

into (64), the Dirac equation becomes

$$\left[\frac{d}{d\eta} - i\alpha \cdot \mathbf{k} + iam\gamma^0\right]\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix} = 0, \tag{67}$$

which reduces to the coupled equations

$$\left[\frac{d}{d\eta} + ima\right]\Phi_1 - i\sigma \cdot \mathbf{k}\Phi_2 = 0, \tag{68}$$

$$\left[\frac{d}{d\eta} - ima\right]\Phi_2 - i\sigma \cdot \mathbf{k}\Phi_1 = 0, \tag{69}$$

inserting

$$\Phi_2 = -\frac{i\sigma \cdot \mathbf{k}}{k^2}\left[\frac{d}{d\eta} + ima\right]\Phi_1, \tag{70}$$

into (69) we arrive at

$$\left[\frac{d^2}{d\eta^2} + m^2a^2 + im\dot{a} + k^2\right]\Phi_1 = 0. \tag{71}$$

We denote positive and negative frequency solutions respectively by $\Phi_{1(2)}^{(+)}$ and $\Phi_{2(1)}^{(+)*}(\Phi_{1(2)}^{(-)})$. Bogolubov transformation between *in* and *out* modes are

$$\Phi_1^{in(+)} = \alpha_k\Phi_1^{out(+)} + \beta_k\Phi_2^{out(+)*}, \tag{72}$$

$$\Phi_2^{in(-)} = \Phi_1^{in(+)*} = \alpha_k^*\Phi_1^{out(+)*} + \beta_k^*\Phi_2^{out(+)}, \tag{73}$$

by taking $\eta \to +\infty$ we arrive at

$$\Phi_1^{in(+)} \sim \alpha_k e^{-ik\eta} + \beta_k e^{ik\eta}, \tag{74}$$

$$\Phi_2^{in(-)} \sim \alpha_k^* e^{ik\eta} + \beta_k^* e^{-ik\eta}. \tag{75}$$

Then the rate of particles and antiparticles created are the same. For scale factor (47) in the case $s = -2$, positive *in*-mode solution will be

$$\Phi_1^{in(+)}(\eta) = \omega^{-1/2}(1 + e^{\lambda\eta})^{\frac{im}{\lambda}} e^{-i\omega\eta} \times F\left(-\frac{i}{\lambda}(-m + \omega + k), \frac{i}{\lambda}(k + m - \omega), 1 - \frac{2i\omega}{\lambda}, -e^{\lambda\eta}\right), \tag{76}$$

where we have introduced

$$\omega = [k^2 + m^2]^{\frac{1}{2}}. \tag{77}$$

By using the relation (58) the Bogolubov coefficients are given by

$$\alpha_k = \left(\frac{k}{\omega}\right)^{1/2}\frac{\Gamma(1 - 2i\frac{\omega}{\lambda})\Gamma(-2i\frac{k}{\lambda})}{\Gamma(-\frac{i}{\lambda}(-m + \omega + k))\Gamma(1 + \frac{i}{\lambda}(-m - k - \omega))}, \tag{78}$$

$$\beta_k = \left(\frac{k}{\omega}\right)^{1/2}\frac{\Gamma(1 - 2i\frac{\omega}{\lambda})\Gamma(2i\frac{k}{\lambda})}{(\frac{i}{\lambda}(k + m - \omega))\Gamma(1 + \frac{i}{\lambda}(k - m - \omega))}. \tag{79}$$

From above and taking into account normalization condition for Dirac field $|\alpha_k|^2 + |\beta_k|^2 = 1$ we arrive at particle (n_p) and antiparticle (n_a) creation rates as follows

$$n_p = n_a = \frac{\gamma_k}{1+\gamma_k}, \tag{80}$$

where $\gamma_k = |\beta_k|^2/|\alpha_k|^2$. With using relations (78) and (79) we obtain

$$\gamma_k = \frac{(m-k+\omega)(m-k-\omega)\sinh\frac{\pi}{\lambda}(m-k+\omega)\sinh\frac{\pi}{\lambda}(m+k-\omega)}{(m+k+\omega)(m+k-\omega)\sinh\frac{\pi}{\lambda}(m+k+\omega)\sinh\frac{\pi}{\lambda}(m-k-\omega)} \tag{81}$$

$$= \frac{\cosh\frac{2\pi}{\lambda}m - \cosh\frac{2\pi}{\lambda}(k-\omega)}{\cosh\frac{2\pi}{\lambda}(k+\omega) - \cosh\frac{2\pi}{\lambda}m}. \tag{82}$$

Then the density of total created particles is

$$n = n_a + n_p = \frac{\cosh\frac{2\pi}{\lambda}m - \cosh\frac{2\pi}{\lambda}(k-\omega)}{\sinh\frac{2\pi}{\lambda}k\,\sinh\frac{2\pi}{\lambda}\omega}. \tag{83}$$

For very light particles we arrive at the following distribution

$$n \approx \frac{2\pi^2 m^2}{\lambda^2}\left(\sinh\frac{2\pi}{\lambda}k\right)^{-1} + O(m^3). \tag{84}$$

Then the total number of created particles is

$$N = \int_0^\infty n d^3k \approx \frac{\lambda m^2 \pi^2}{12}. \tag{85}$$

For massless neutrinos, as a result of conformal invariant, particle creation rate is zero.

4 Conclusion

In curved spacetimes when there are no asymptotically static *in* and *out* regions we can not define particle and vacuum state like Minkowskian space. In particular when universe expand from a singular origion, the notion of an initial vacuum is ambiguous. In this paper we introduce the spacetime with the scale factor $(\frac{1+\delta}{2} + \frac{1-\delta}{2}\tanh(\lambda\eta))^s$ which is manifestly Minkowskian in the remote past and future. In this model of spacetime *in* and *out* vacua are well defined because the scale factor reduces to a constance at asymptotic past and future times, therefore in this regions there is a timelike killing vector and one can defined particle state in terms of positive frequency modes. We show that massive conformally coupled scalar field is exactly soluble for $s = \pm 1, \pm 2$. With a straightforward calculation of the Bogolubov coefficients between the *in* and *out* modes, the rate of particles production during the expansion of the universe is obtained. We show that the density of particles created for high masses is thermal. Another soluble model is $(1 + e^{\lambda\eta})^s$, which has an asymptotically static *in* region. In this model positive and negative frequency modes in the remote past are well defined, but in *out* region we used WKB approximation method for definition of positive frequency modes. Massive conformally coupled scalar field is soluble for $s = \pm 1, \pm 2$. We considered to case $s = -2$, in this case we can solve exactly massive minimally coupled scalar field as well as massive conformally coupled scalar field. We show that for $s = -2$ in asymptotically regions, conformal time and cosmological time are equivalent. After solution of scalar field equation we obtain the particles creation rate. We also investigate creation of massless minimally coupled scalar field. Also we found the analytic solutions of Dirac equation, which is important because finding exact solutions of Dirac equation on given backgrounds is not easy always. We found the solutions which behaves as positive frequency modes in the asymptotic regions. We have shown that the rate of creation for particles and antiparticles are equal, which is a favorable result in cosmology indicating creation of equal amount matter and antimatter.

References

1. Parker, L.: Phys. Rev. Lett. **21**, 562 (1968)
2. Parker, L.: Phys. Rev. **183**, 1057 (1969)
3. Parker, L.: Phys. Rev. D **3**, 346 (1971)
4. Parker, L.: Phys. Rev. D **3**, 2546 (1971)
5. Birrel, N.D., Davies, P.C.W.: Quantum Fields in Curved Space. Cambridge Univ. Press, Cambridge (1982)
6. Linde, A.D.: Particle Physics and Inflationary Cosmology. Harwood Academic, Reading (1990)
7. Kolb, E.W., Turner, M.S.: The Early Universe. Addison-Wesley, Reading (1990)
8. Barut, A.O., Duru, I.H.: Phys. Rev. D **36**, 3705 (1987)
9. Isham, C.J., Nelson, J.E.: Phys. Rev. D **10**, 3226 (1974)
10. Villalba, V.M., Percoco, V.: J. Math. Phys. **31**, 715 (1990)
11. Bernard, C., Duncun, A.: Ann. Phys. **107**, 201 (1977)
12. Abramowitz, M., Stegun, I.: Handbook of Mathematical Functions. Dover, New York (1974)